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Natural frequencies of bars with variable cross-sections

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Abstract: This paper presents the possibility of determining the natural frequency of flexural bars treated as Bernoulli-Euler beams using the Rayleigh method. It is assumed that the shape of the bar axis during vibration is the same as the shape of the deflection line of this bar under continuous load. By comparing the potential energy in the deflected position with the kinetic energy in the undeflected position, it is possible to determine the frequency and period of natural vibration as a function of Young's modulus and material density, as well as the parameters describing the geometric shape of the bar. Examples of solutions for truncated cone and truncated wedge-shaped bars are shown, as well as a solids of revolution with the generatrix described by linear and curvilinear (parabolic, exponential) forms. It applies to both solid and hollow bars, and different types of fixing the ends of the bar. The accuracy of the obtained results (compared with the literature and with the results of FEM calculations) is sufficient for practical applications. The authors also showed the possibility of extending the method to higher frequencies of vibration.

Keywords: natural frequency, bar with variable cross-section, Rayleigh method

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1. Introduction

Variable cross-section bar elements are often used to reduce the weight of structures. This applies both to machine parts and elements of building structures. The natural frequencies of the bars can be determined using the differential equation of a Bernoulli-Euler beam. Conway and Dubil [1], obtained an analytical solution of this equation using the Bessel function for the cases of truncated cone and truncated wedge. The solution of this equation by the Frobenius method is given in [2]. The author has included tables with the values of the first three natural frequencies for 16 combinations of support schemes and different values of the parameter determining the convergence of the walls of the truncated cone and truncated wedge. Many researchers have extended natural vibration analyses to bars with other cross-section shapes, often obtaining solutions only for special cases. For example, [3] studied the natural vibration of a bar with exponentially varying cross-sectional width, and [4] analyzed a bar in the shape of a solid of revolution with a radius changing parabolically along the length of the bar. The subject of the study of the paper [5] was a bar in the shape of a hollow truncated cone. The paper [6] analyzed the vibration of a bar in the shape of a solid of revolution with generatrix described by exponential and trigonometric functions. A variety of methods for analyzing the vibration of beams with variable cross-sections can also be found in other papers, such as [7, 8]. The paper [9] deals with solid and hollow wedge-shaped bars of varying cross-sections with additional point masses. Beams with variable cross-sections or with concentrated masses can also be analyzed as multi-section bars with stepped cross-sections [10, 11].

The authors of this paper use the Rayleigh method to determine the first natural frequency of flexural bars treated as Bernoulli-Euler beams. It is assumed that the shape of the bar axis during vibration is the same as the shape of the deflection line of this bar under a continuous load of constant value. For a perfectly elastic material and a small amplitude of vibration, the potential energy in the deflected position is compared with the kinetic energy in the undeformed position. This allows the determination of the frequency and period of natural vibration as a function of Young's modulus and density of the material, as well as parameters describing the geometric shape of the bar. Detailed solutions using this method have been published in a number of papers, including in [12–15].The aim of this paper is to collect in one place information about the possibilities of using this method, show selected results of calculations published in various own papers and provide a critical review of the usefulness of the method used by the authors.

2. Method

The procedure using the Rayleigh method [12] in the way adopted by the authors of this paper can be explained using the example of a cantilever bar in the shape of a truncated cone of length L and wall convergence $\eta = D/d$ (Fig. 1).

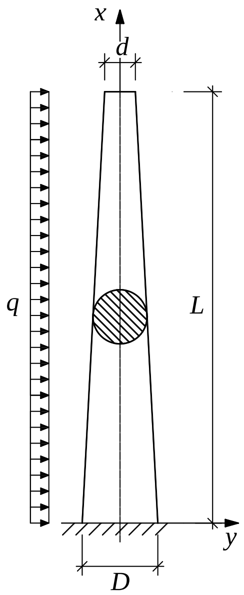

Fig. 1. Scheme of the cantilever bar.

The deflection of the bar was determined by integrating the differential equation of the deformed axis twice:

$$EJ\left(x\right)\frac{d^2u\left(x\right)}{dx^2} = -M\left(x\right) = \frac{qL^2}{2} - qLx + \frac{qx^2}{2} \tag{1}$$

where:
E – Young's modulus,
J – moment of inertia of the section,

u – deflection,
M – bending moment in the corss-section defined by the x coordinate,
q – continuous load with a constant value.

The deflection line of the bar as in Fig. 1 is defined by the equation:

$$u(x) = \frac{32q}{\pi E a^4} A(x) \tag{2}$$

where:

$$a = \frac{d - D}{L}$$

$$A(x) = \frac{3D(D-d)+d^2}{3D^3} ax + \frac{d(6D-d)}{6D^2} + \ln\frac{D}{ax+D} - \frac{d}{ax+D} + \frac{d^2}{6(ax+D)^2}$$

The potential energy in a deflected position is given by the relation:

$$E_p = \int_0^L \frac{1}{2} q u(x) dx = \frac{1}{2} q \frac{32q}{\pi E a^4} \int_0^L A(x) dx \tag{3}$$

and kinetic energy is equal:

$$E_k = \int_0^L \frac{1}{2} \omega^2 u^2(x) m(x) = \frac{1}{2} \omega^2 \frac{\pi\rho}{4} \left(\frac{32q}{\pi E a^4} \right)^2 \int_0^L (ax+D)^2 A^2(x) dx \tag{4}$$

where:
ω – vibration frequency,
ρ – material density,
$m(x)$ – mass of a material slice with a thickness dx.

The energy comparison enables to determine the frequency in the form:

$$\omega = \frac{(\eta-1)^2 d^2}{2L^2} \sqrt{\frac{E}{2\rho}} \sqrt{\int_0^L A(x) dx \cdot \left(\int_0^L (ax+D)^2 A^2(x) dx \right)^{-1}} \tag{5}$$

and after integration and ordering with the use of the Mathematica program, the relation was obtained:

$$\omega^2 = \frac{E}{\rho} \cdot \frac{45D^2(\eta-1)^4}{2L^4\eta^2} \cdot \frac{Q_3}{Q_1 + 60Q_2 \ln\eta} \tag{6}$$

where:

$$Q_1 = 11\eta^6 - 87\eta^5 + 375\eta^4 - 1184\eta^3 + 3219\eta^2 - 4281\eta + 2817 - 1080\eta^{-1} + 234\eta^{-2} - 24\eta^{-3}$$

$$Q_2 = 14\eta^3 - 33\eta^2 + 15\eta - 3 - 6\eta^3 \ln\eta$$

$$Q_3 = 3\eta^4 + 10\eta^3 - 18\eta^2 + 6\eta - 1 - 12\eta^3 \ln\eta$$

Similarly proceeding, solutions have been obtained for cantilever bars in the form of a truncated wedge [12], solid and hollow solids of revolution with a genetratrix in form of parabola, and an exponential curve [13, 16].

Another, more universal approach to solving the same problem is to quadratically integrate the differential equation of the deformed axis in the form:

$$\frac{d^2}{dx^2}\left(EJ(x)\frac{d^2u(x)}{dx^2}\right)=q \tag{7}$$

The integration constants are determined from the boundary conditions, making it possible to obtain the equation of the deflection curve for each of the analyzed cases of the beam support scheme, and after determining and comparing the energy, determine the natural frequency. The paper [14] considers six supporting schemes: clamped – pinned, clamped – sliding, clamped – clamped, clamped – free, pinned – sliding, pinned – pinned. It analyzed bars in the shape of a truncated cone and a truncated wedge.

For a simply supported bar in the shape of a truncated wedge, the relation for the first natural frequency after some transformations can be given in the following form, [17]:

$$\omega^2=\frac{E}{\rho}\cdot\frac{72H^2(\phi-1)^5}{L^4}\cdot\frac{3-3\phi^2-\left(1+4\phi+\phi^2\right)\ln\phi^{-1}}{R_1+R_2\ln^2\phi^{-1}+R_3\ln\phi^{-1}} \tag{8}$$

where:

$\phi=\dfrac{h}{H}$

$R_1=551\phi^5-783\phi^4+232\phi^3+232\phi^2-783\phi+551$

$R_2=72\phi^5+360\phi^4+648\phi^3+648\phi^2+360\phi+72$

$R_3=396\phi^5+744\phi^4-312\phi^3+312\phi^2-744\phi-396$

h and *H* are the smallest and largest heights of a wedge of length *L*.

The procedure outlined above requires integration of complex expressions. In the case of solid bars (and some special cases of hollow bars) in the shape of a truncated cone and a truncated wedge, simple solutions are obtained and a pocket calculator is enough for calculations. In many more complicated cases, numerical integration is required. In the method described by the authors, integration can be replaced by calculations performed for the individual sections of the bar into which it has been divided. If the number of sections is very large, the results approach the exact solution obtained by integration. The paper [18] presents an algorithm developed by the authors that performs such calculations in the Mathematica environment. This program can be used for bars of several sections, where the shape of each section is described by a different equation, and concentrated masses can be taken into account in this program.

3. Examples

The authors obtained solutions for cantilever bars in the shape of a truncated cone and solids of revolution with generatrix described by curvilinear (parabolic, exponential) forms. This applies to both solid and hollow bars. Figure 2 shows a comparison of the vibration periods of solid steel columns (L = 6 m, D = 0.2 m) in the shape of solids of revolution with generatrix described by a straight line, a parabola and an exponential function. The full range of bar wall convergence was considered, η = 1 (cylinder) to $\eta = \infty$ (cone). The results were compared with the results of FEM calculations in ANSYS. It can be seen that even small differences in the shape of the column with a generatrix described by a parabola and an exponential function lead – especially at large convergences of the bar walls, defined as $\eta = D/d$ – to significant differences in the values of vibration periods.

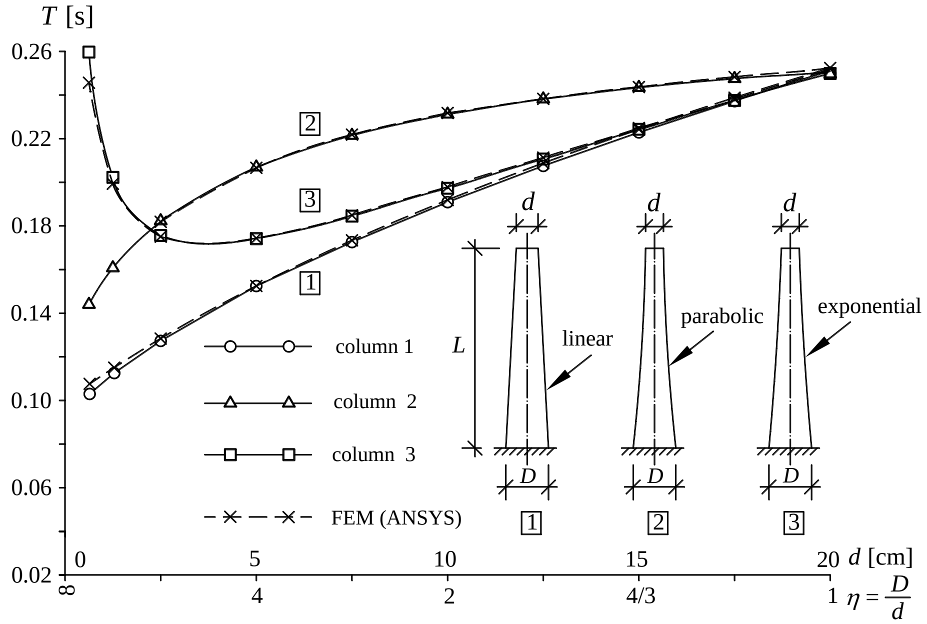

Fig. 2. Periods of vibration of solid steel columns in the shape of solids of revolution with linear, parabolic and exponential generatrix, [16].

The calculation method applies to elastic materials and can be used for reinforced concrete structures, for example, towers with a reinforced concrete shell structure or chimneys as in Fig. 3, where L = 32 m, D = 4 m and the wall thickness (in the plane perpendicular to the axis of the chimney) varies linearly from 0.24 m at the base to 0.12 m at the top.

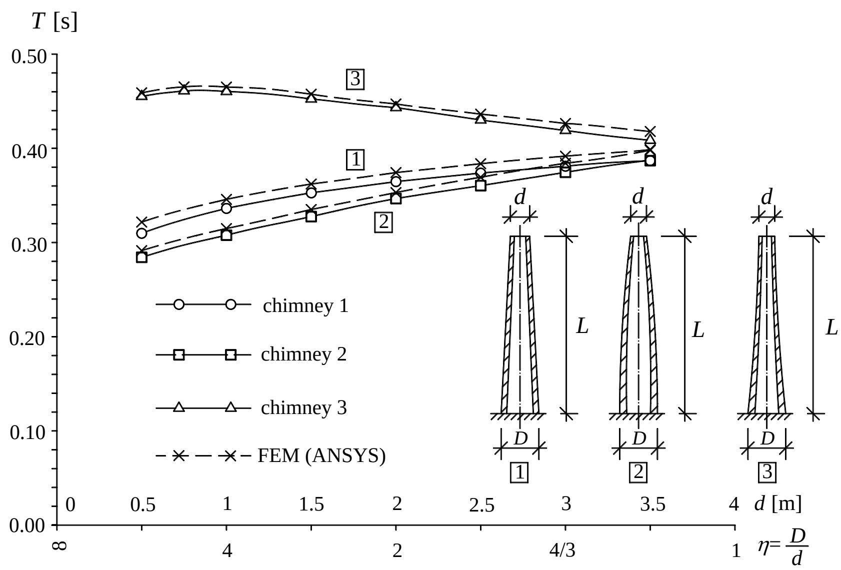

Fig. 3. Periods of vibration of reinforced concrete chimneys with linear and parabolic generatrix (convex and concave parabola with respect to the cone axis), [13, 16].

Figure 4 shows the result of using the program prepared in the paper [18] when applied to a simply supported symmetric beam with straight haunches in the support zones. The material was treated as homogeneous and perfectly elastic, and its density and longitudinal modulus of elasticity were assumed. In the sections at the supports and in the middle part, the height of the beam is fixed, L = 10 m, H = 0.8 m, different values of h and proportions of the L_1 and L_2 sections were taken into account. Knowledge of the natural frequency of floor beams is important for checking Serviceability Limit State (SLS).

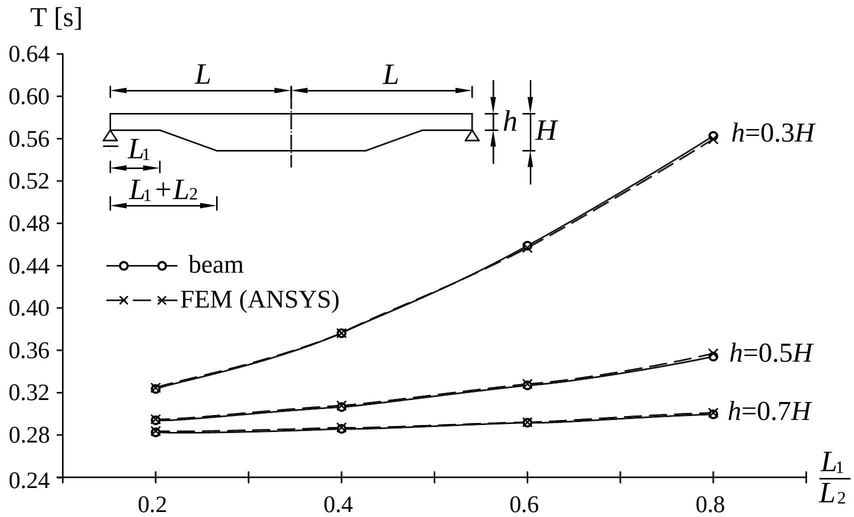

Fig. 4. Periods of natural vibration of reinforced concrete floor beam, [18].

4. Higher natural frequencies

A simply supported beam in the shape of a truncated cone as shown in Figure 5a was considered.

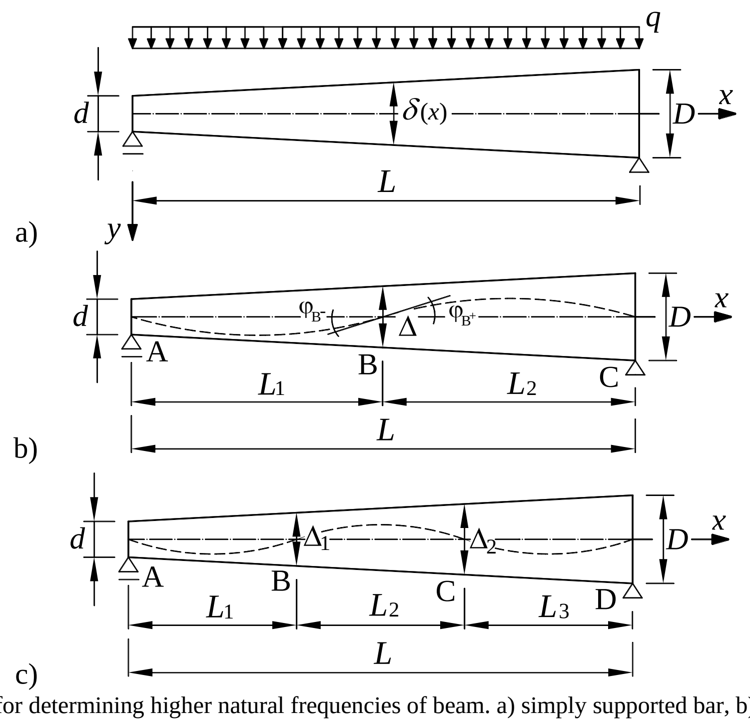

Fig. 5. Scheme for determining higher natural frequencies of beam. a) simply supported bar, b) scheme to determine an inflection point of the deflection curve for the second natural frequency, c) scheme for the third natural frequency, [17].

The differential equation of the deformed axis has the form:

$$EJ(x)\frac{d^2u(x)}{dx^2}=\frac{q}{2}\left(x^2-Lx\right) \tag{9}$$

After the integration, from the comparison of the potential and kinetic energies of the vibrating bar, the relation for determining the first natural frequency is obtained in the form, [17]:

$$\omega_1^2=\frac{E}{\rho}\cdot\frac{45D^2(\psi-1)^5\psi}{2L^4}\cdot\frac{P_1-6\psi(1-\psi)\ln\psi}{P_2-3\psi\ln\psi\left(P_3-P_4\ln\psi\right)} \tag{10}$$

where:

$\psi=\dfrac{d}{D}$

$P_1=\psi^3+9\psi^2-9\psi-1$

$P_2=4\psi^8+9\psi^7+36\psi^6+311\psi^5-720\psi^4+311\psi^3+36\psi^2+9\psi+4$

$P_3=8\psi^6+13\psi^5+80\psi^4-80\psi^2-13\psi-8$

$P_4=12\psi\left(1+\psi+\psi^2+\psi^3+\psi^4\right)$

Then the relations for the slop angle of the left end (for x = 0) and for the right end (for $x = L$) of the beam were determined. In such a defined system of coordinate axes, the value of the slope angle at the left end of the beam is positive, at the right end is negative. Writing down these relations for both sections of the beam (Figure 5b) and comparing these angles (with due regard to their signs) on the left and right sides of the inflection point B of the deflection curve leads to relation (11), from which it is possible to determine the positions of point B, determine the values of diameter Δ and then the lengths of sections L_1 and L_2.This allows comparing the potential and kinetic energy of the two sections and determining the second natural frequency ω_2.

$$\begin{aligned}&-\frac{D-\Delta}{\Delta-d}\left[\ln\frac{d}{\Delta}+\frac{(\Delta-d)(\Delta+4d)}{3d\Delta}-\frac{(\Delta-d)(3\Delta+d)}{6\Delta^2}\right]=\\&=\left[\ln\frac{\Delta}{D}+\frac{D^2-\Delta^2}{3\Delta D}+-\frac{(D-\Delta)(3\Delta-D)}{6\Delta^2}\right]\end{aligned} \tag{11}$$

The same procedure can be done during determining higher natural frequencies, as shown in the example of the third frequency. A bar of length L was divided into three parts (Figure 5c), from the condition of equality of the slop angles of the section on both sides of the bar at points B and C, a system of two equations was obtained. Using Mathematica, the values of Δ_1 and Δ_2 were determined and then L_1, L_2 and L_3. This allowed the energies to be calculated and the third natural frequency to be determined.

In order to determine the n-th natural frequency, the bar must be divided into n sections, which allows to obtain n – 1 equations from which the diameters of the bar at the ends of each section and the lengths of these sections can be determined, and from a comparison of energies, the n -th natural frequency can be determined.

The differences between the first natural frequency of simply supported bars according to the tables in the paper [2] and the results of FEM calculations and the results obtained by the authors do not exceed 0.6% for truncated cone-shaped bars and 0.3% for truncated wedge-shaped bars. For the second and third natural frequencies, results with small error were obtained for truncated cone-shaped bars with small walls convergences $\psi = d/D \geq 0.6$ (differences less than 5%) and for truncated wedge-shaped bars for $\varphi = h/H \geq 0.4$ (differences less than 6%). For bars with larger walls convergences, the results are no longer accurate enough.

5. Conclusions

The application of the Rayleigh method to determine the first natural frequency of the bar, assuming that the axis of the bar deformed during vibration takes the form of an axis deflection line under continuous load with constant value, leads to acceptably approximate results. As shown in [12], changing the load from a continuous load of constant value to a continuous load varying proportionally to the diameter or cross-sectional area of a truncated cone-shaped bar does not lead to an increase in the accuracy of the results.

For truncated cone and truncated wedge-shaped bars, the frequency (or period) of vibration can be represented by simple formulas, and a pocket calculator will be enough. The accuracy of the results for truncated cones and truncated wedges is high. The differences between the frequencies (or periods) of vibration calculated by the method used by the authors and the results of the exact solutions did not exceed 0.5% for simply supported beams in the shape of a truncated wedge and 1.7% for beams in the shape of a truncated cone, while 0.6% for simply supported beams.

For bars in the shape of solids of revolution with curvilinear generatrix and for hollow bars, solutions using the method applied by the authors require numerical integration, so they cannot be presented in the form of elementary formulas, and it is necessary to use computational programs for calculations. The accuracy of the calculations is lower. The point of reference here is the calculation made by the finite element method, the results of which are also subject to some error. In the examples analysed by the authors, for bars in the shape of hollow truncated cones, the differences between the obtained results and FEA results did not exceed 2.6%. For bars in the shape of solids of revolution with generatrix described by a parabola (concave or convex with respect to the cone axis) or an exponential function, the largest differences did not exceed 5.6% with respect to FEM solutions. This is enough accuracy for most practical engineering calculations.

In the case of the second and third natural frequencies of simply supported bars in the shape of a truncated cone and a truncated wedge, the accuracy of the results significantly depends on the degree of convergence of the side walls of the bars. The differences between the first natural frequency of simply supported bars according to the tables in the paper [2] and the results of FEM calculations and the results obtained by the authors do not exceed 0.6% for bars in the shape of a truncated cone and 0.3% for bars in the shape of a truncated wedge. In contrast, for the second and third vibration frequencies, differences of less than 5% were obtained for truncated cone-shaped bars with small walls convergences ($\psi \geq 0.6$), and differences of less than 6% for truncated wedge-shaped bars with small walls convergences ($\varphi \geq 0.4$). At higher bar walls convergences, the results become less accurate. In addition, calculations for the second, third and higher vibration frequencies are increasingly laborious.

In summary, it should be said that the method described by the authors of this paper is accurate for the first natural frequency. At the same time, for bars in the shape of a truncated cone and a truncated wedge, the solutions are very simple, while for hollow bars and bars in the shape of solids of revolution with curvilinear generatrix, the method becomes laborious.

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